CCM-79-18

Center for Composite Materials

NOTCHED STRENGTH PREDICTION OF COMPOSITE MATERIALS USING WEIBULL-TYPE INTEGRALS FOR UNIAXIAL AND BIAXIAL LOADING

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October 1979

Revised December 1979

Abstract

The Weibull distribution model for brittle fracture is applied to a hole size study in uniaxial tension and to a study in biaxial tension. This application of the Weibull model uses a numerical integration of stress functions across a high-failure-risk volume. The results for uniaxial tension indicate a hole size effect which agrees in form with the "point-stress" or characteristic dimension theory. The uniaxial strength predictions based on Weibull theory are uniformly conservative. The predictions also show increasing notch sensitivity as the material becomes more "perfect", that is, has fewer and smaller inherent flaws. The biaxial Weibull study accurately predicts the failure mode and the strengthening effect of biaxial tension. The biaxial strength predictions are also generally conservative.

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Author's Note on Weibull Statistics

It is important for the reader to realize the difference between the use of Weibull statistics for curve fitting and the calculation of "Weibull-type integrals."

The Weibull distribution may be used purely as a curve-fit scheme. In this use, it has general utility, much as a normal distribution or any other distribution for describing data scatter. This paper, however, uses a much more specific application, that of "Weibull-type integrals." In this application, the parameters of the Weibull distribution have physical significance; they are related to stress and geometry variables. These integrals attempt to calculate parameters for a Weibull distribution, rather than simply infer them from data as in a curve-fit scheme.

Introduction

The problem of strength degradation in the presence of a notch is important to the materials designer. These notches may be introduced intentionally, as in the case of bolt holes or cutouts, or unintentionally, as in the case of a materials fabrication error or a fatigue crack. The designer needs formulae which help him estimate strength "knock-down factors" for his design; he also needs a way of comparing the notch sensitivity of different materials. Predictions of notch sensitivity at a variety of hole sizes based on the finite state of stress in the material have been successful, [Pipes, Wetherhold, Gillespie],[Pipes, Gillespie, Wetherhold]. This model is known also as the "point-stress" model. The meaning of the notch sensitivity parameters found in the above references has not, however, been related to any specific physical variables.

Another important factor for a materials designer is to understand how the inherent variability of his material will affect design. Factors of reliability must often be calculated based on a number of strength data to insure a certain low probability of failure. Weibull statistics are often employed as a good way of calculating reliability.

This paper joins the strength degradation predictions of the point-stress model to the use of Weibull statistics. In this way, the parameters contained in the point-stress strength equations are related to the unnotched tensile strength data scatter. Thus, the variability in unnotched strength is related to the strength degradation in the presence of a notch or flaw. The specific calculations in this paper are for both through-holes and slits (notches) in a uniaxial tension plate, and for through-holes in a biaxial tension plate.

Part 1 UNIAXIAL TENSION

1.1 Weibull Theory

The Weibull brittle failure model may be used to predict conservative failure loads for notched composites. The prediction is based on an integration which, ideally, would be over the entire stressed volume. For a plane stress state of a notched plate with one predominant stress direction "x," the stress state may be given by

$$\sigma_x(x,y) = \sigma_x^{\infty}\, t(x,y) \qquad 1.1$$

where σ_x^{∞} is the far field stress, and $t(x,y)$ is a geometric locus. [see Appendix B]. See Fig. 1a. In general, $t(x,y)$ would come from a complex variable solution for stress. Fig. 1a' shows an elliptic center through-notch, although the actual notch could be any shape.

Using the results of Appendix A, we note that the average far-field failure stress can be found as follows. The stress integral used in reliability calculations is

$$B(\sigma_x^{\infty}) = \int_V \left(\frac{\sigma}{\beta}\right)^{\alpha} dV \qquad 1.2$$

where we assume that $\sigma = \sigma_x$ is the predominant stress. Since equation 1.1 introduces the geometric variation in stress, we may write

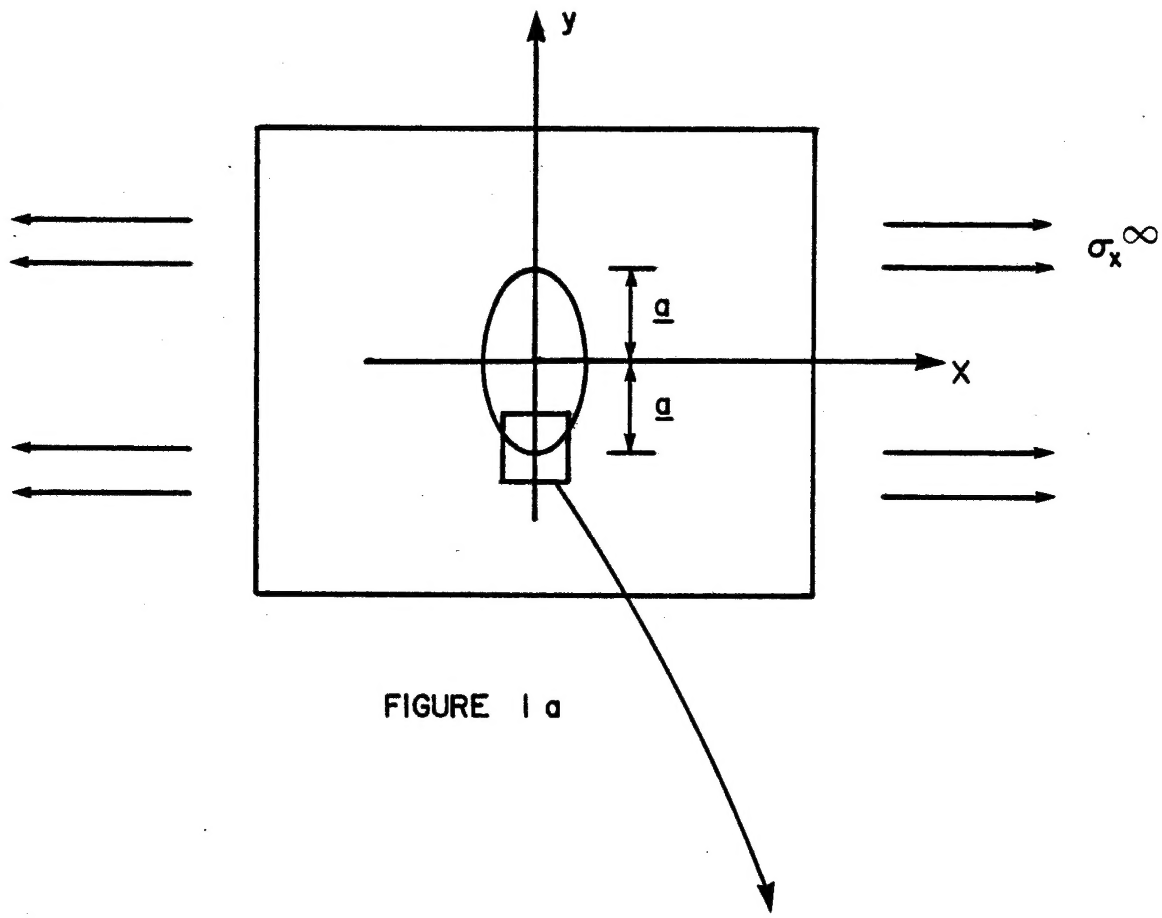

FIGURE 1a

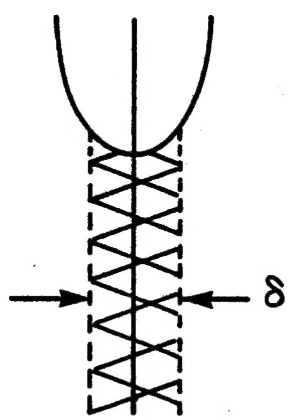

FIGURE 1b

$$\int_V \left(\frac{\sigma_x}{\beta}\right)^{\alpha} dV = \int_V \left(\frac{\sigma^{\infty} t(x,y)}{\beta}\right)^{\alpha} dV \qquad 1.3$$

Note that σ^{∞} may be removed from the integral. In order to place equation 1.3 in a standard form, we define a new notch scale parameter (dependent on notch geometry) by

$$\beta_N = \left(\int_V [t(x,y)^{\alpha}] dV \right)^{-1/\alpha} \beta \qquad 1.4$$

This way, the reliability function R for probability of survival at any given far-field stress σ_x^{∞} is

$$R(\sigma^{\infty}) = \exp\left[-\left(\frac{\sigma^{\infty}}{\beta_N}\right)^{\alpha} \right] \qquad 1.5$$

or $$R(\sigma^{\infty}) = e^{-B(\sigma^{\infty})} \qquad 1.6$$

The average value of far-field stress, the most meaningful statistic from small quantity notched strength tests, can now be calculated. From Appendix A.I, the average notched strength $\overline{\sigma}_N$ is related to the scale parameter β_N by

$$\overline{\sigma}_N = \beta_N \, \Gamma(1+1/\alpha) \qquad 1.7$$

We now hypothesize that the volume of integration can be reduced to a thin strip perpendicular to the applied loading. See Fig. 1b. The strip has width "δ" by plate

thickness "h." Within the strip $-\frac{\delta}{2} \leq Y \leq \frac{\delta}{2}$, $t(x,y) \simeq t(x,0)$ (stress within the volume ≃ stress at the line $Y = 0$); $\delta \leq .05\,\underline{a}$ is sufficiently small for error to be ≤ 1%. The strip length could be from $Y = \underline{a}$ to the plate edge (finite plate), or to 10 times maximum $\underline{a}$ (infinite plate). The result of this narrowing of the integration volume of interest is to reduce equation 1.4 to

$$\beta_N = \left(h\delta \int_{\underline{a}}^{L} [t(0,y)]^{\alpha} \, dY \right)^{-1/\alpha} \beta \qquad 1.8$$

We would expect such a calculation to give conservative results, as the volume under consideration is the highest risk vlume. The average predicted failure load can be found by combining equation 1.8 with 1.7,

$$\boxed{\overline{\sigma}_N = \left(h\delta \int_{\underline{a}}^{L} [t(0,y)]^{\alpha} \, dY \right)^{-1/\alpha} \beta\Gamma(1+1/\alpha) \qquad 1.9}$$

In order to calculate the notched/unnotched strength ratio, we make the following assumption:

> α, the shape parameter,is identical
> for the notched and unnotched specimens.

The average strength for an unnotched specimen, using the same volume of integration as for the notched specimen (with $\underline{a} = 0$) is

$$\overline{\sigma}_o = [h\delta L]^{-1/\alpha} \beta\Gamma(1+1/\alpha) \qquad 1.10$$

This leads to an equation predicting the reduction in strength with a notch; dividing equation 1.9 by 1.10,

$$\frac{\sigma_N}{\sigma_o} = \frac{\left(\int_{\underline{a}}^{L} [t(0,Y)]^{\alpha} dY \right)^{-1/\alpha}}{L^{-1/\alpha}} \qquad 1.11$$

where we omit the super-bar, but define the lefthand side as average values. Note that as $\underline{a} \to 0$, $\sigma_N/\sigma_o \to 1$ as it should. We are now in a position to vary the parameters in equation 1.11, evaluating the integral 1.11 numerically.

1.2 Point Stress Theory

This failure theory is introduced by manner of comparison with Weibull theory. The size effect predicted by point stress theory is, in fact, identical in form to that of Weibull theory.

The loading geometry is identical to Fig. 1. The point stress failure prediction is that when the stress at some distance "d_o" into the material reaches the unnotched ultimate stress, the sample fails. i.e. failure occurs when

$$\sigma \Big|_{\underline{a}+d_o} = \sigma_o \qquad 1.12$$

where σ_o is the unnotched ultimate [Whitney, Nuismer]. To provide an absolute hole size effect in agreement with experiment to vary with notch size per [Pipes, et.al.]

$$d_o = (\underline{a}/\underline{a}_o)^m/K \qquad 1.13$$

where $\underline{a}$ = notch half-length

$\underline{a}_o$ = one times units used (arbitrary)

K = notch sensitivity parameter

m = exponential parameter

Since the direction of fracture is presumed perpendicular to the applied load, only the stress formula for $\sigma(x=0,y)$ is required. The reference notch half size $\underline{a}_o$ is included only to avoid awkwardness in the units of K.

1.3 Numerical Integral Results

The stress profile in the selected volume of integration is complex enough that equation 1.11 demands numerical integration. The integration routine selected is DCADRE, a part of the International Mathematical and Scientific Library (IMSL).

1.3.1 Circular Hole

For a circular through hole geometry, we have the infinite plate stress profile [Konish, Whitney]

$$t(0,Y) = 1+\frac{1}{2}\rho^{2}+\frac{3}{2}\rho^{4}-(K_T^{\infty}-3)(\frac{5}{2}\rho^{6}-\frac{7}{2}\rho^{8}) \qquad 1.14$$

where $\rho = \frac{R}{Y}$

R = hole radius

K_T^{∞} = classical stress concentration factor, calculatable from the elastic constants

For a finite plate where $\frac{R}{W} < \frac{1}{4}$, we may use an isotropic finite width correction (FWC) and multiply the results of equation 1.14 by the FWC factor [Peterson]

$$FWC = 1 - 0.05\ \lambda + 1.5\ \lambda^{2} \qquad 1.15$$

$\lambda = 2R/W$

W = sample width

A hole size series was run in which the specimen width was held constant, and the hole radius was varied at a given Weibull shape parameter α. The material being modeled was isotropic; thus, $k_T^\infty = 3$.

The resulting σ_N/σ_o values were used to back-calculate a d_o value. Since, for circular notches,

$$\frac{\sigma_N}{\sigma_o} = 1 + \frac{1}{2}\xi^2 + \frac{3}{2}\xi^4 - (k_T^\infty - 3)(\frac{5}{2}\xi^6 - \frac{7}{2}\xi^8) \qquad 1.16$$

where $\xi = \frac{R}{R+d_o}$

this implies an inverse function "g,"

$$d_o = g\left(\frac{\sigma_N}{\sigma_o}\ ;\ R\right) \qquad 1.17$$

The values of d_o calculated from the σ_N/σ_o values were then fitted to the equation 1.13, taking logarithms:

$$\ln d_o = m \ln (\frac{R}{R_o}) - \ln k \qquad 1.18$$

The results, shown in Table 1, demonstrate that the presumed form for d_o as a function of radius (1.13 or 1.18) is extremely accurate.

Increasing α is also seen to bring increased notch sensitivity. Both k and m increase with increasing α, which indicates a more rapid strength degradation in the presence of a

notch. (see Table 1, Fig. 2,3) Physically, a material with high α has a very tight distribution of small flaws. Thus, when we add a notch, it far outweighs any innate flaw effects. Conversely for low α there are various size flaws in the material, and much data scatter. The addition of one more flaw (the notch) is almost unnoticed; notch sensitivity (m and k) are low. Note that the curves in Fig. 2,3 have a "kneeing over"; notch sensitivity reaches a rough constant for α greater than about 30.

Table 1

Weibull Shape Parameter α versus notch sensitivity Constants k and m for Circular Notch

α	m	k	r*
3	0.313	3.89	0.985
5	0.386	5.49	0.987
10	0.520	9.70	0.990
12	0.573	10.75	0.986
15	0.560	14.0	0.983
20	0.611	16.8	0.996*
30	0.572	27.8	0.977*

* correlation coefficient from the least squares fit of linear equation

$$\ln d_o = m \ln \left(\frac{R}{R_o}\right) - \ln k;$$

four points (radii) used for each given α.

** Same as *, except three points (radii) used for each given α.

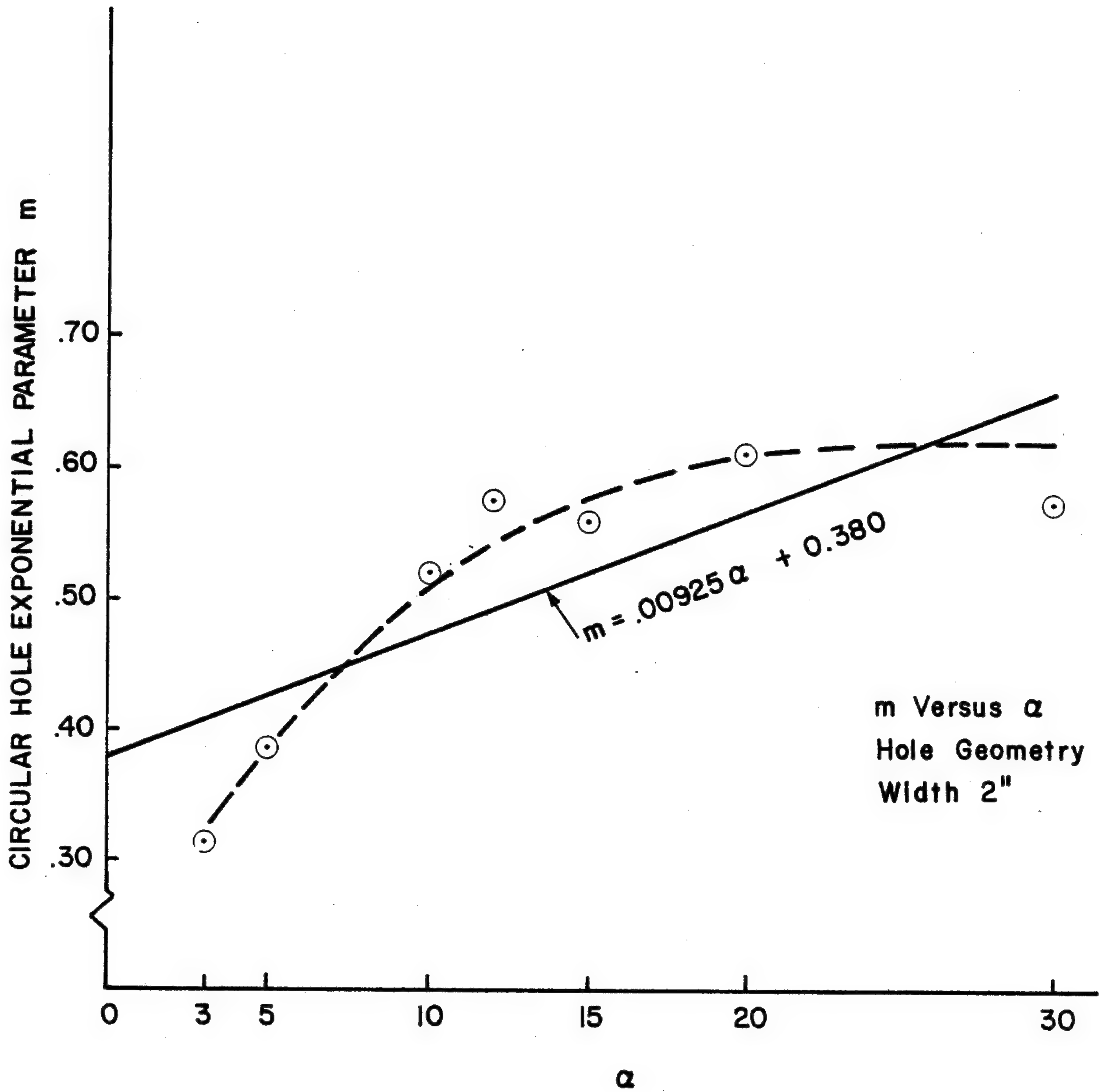

FIGURE 2

m CALCULATED AS A FUNCTION OF α FROM NUMERICAL INTEGRALS

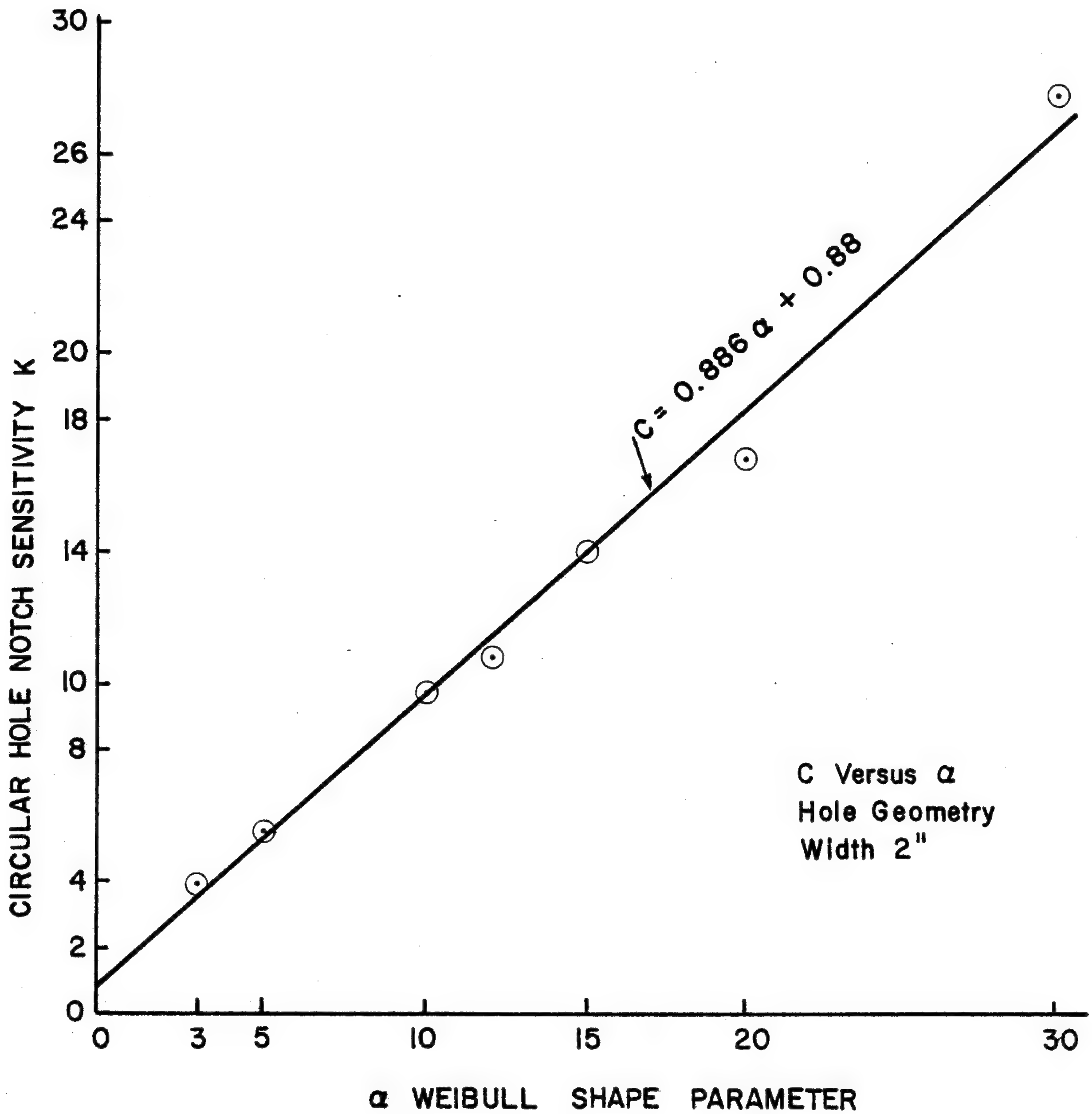

FIGURE 3

k CALCULATED AS A FUNCTION OF α FROM NUMERICAL INTEGRALS

1.3.2 Slit Notch

For a through slit notch (crack), results similar to the hole results were obtained. A degradation of strength with increasing hole size is observed, and the notched strength reduction calculated by Weibull integrals is very conservative.

The theoretical stress profile [Savin] must be modified for use in the numerical integral.* We let

$$t(0,Y) = \begin{cases} \sqrt{Y^2-C^2} & Y \geq 1.01C \\ 7.12 & 1.01c \geq Y \geq C \end{cases} \qquad 1.19$$

The entire profile is multiplied by a finite width correction factor [Paris, Sih]

$$FWC = \left[\frac{W}{\pi C} \tan \frac{\pi C}{W}\right]^{\frac{1}{2}} \qquad 1.20$$

where W = sample width

C = crack half-length

The integrals of equation 1.11 were performed for a given width sample, with the crack sized varying at a given α. The numerical integrals were used to back calculate a value of d_o by inverting the point stress criterion;

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* The IMSL Routine (DCADRE) uses progressively smaller mesh sizes and tries to limit the error between mesh divisions. The theoretical profile is infinite at Y=C; while the integral exists, this form is unsuitable for numerical integration.

since, for cracks,

$$\frac{\sigma_N}{\sigma_o} = \sqrt{1 - \left(\frac{c}{c+d_o}\right)^2} \qquad 1.21$$

thus

$$d_o = C\left[\frac{1}{\sqrt{1 - \left(\frac{\sigma_N}{\sigma_o}\right)^2}} - 1\right] \qquad 1.22$$

The results show that the variation of d_o with crack size,

$$\ln d_o = m \ln(\frac{C}{C_o}) - \ln k \qquad 1.23$$

fits the variation in notch strength extremely well. (See Table 2). The variation in m and k with α may be seen in Figures 4, 5.

Table 2

Weibull Shape Parameter α versus notch sensitivity Constants k and m for Slit Notch

α	m	k	r*
3	0.189	10.9	0.986
5	0.448	23.9	0.999
10	0.723	49.3	0.999
12	0.758	56.8	0.999
15	0.801	64.7	1.000
20	0.836	75.3	1.000

* Correlation coefficient from the last squares fit of the linear form

$$\ln d_o = m \ln (\frac{C}{C_o}) - \ln k;$$

Four points (crack sizes) used for each α.

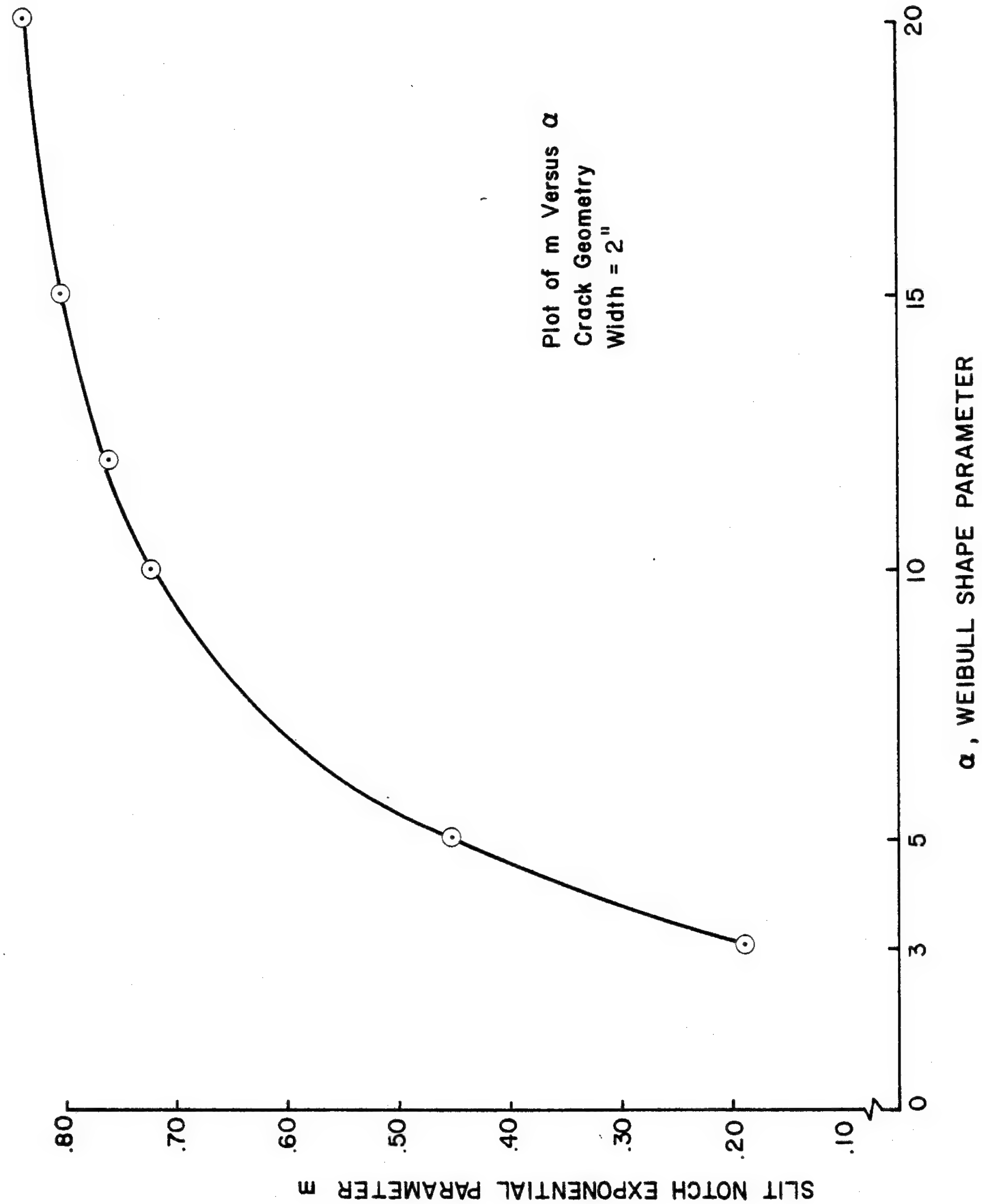

FIGURE 4

m CALCULATED AS A FUNCTION OF α FROM NUMERICAL INTEGRALS

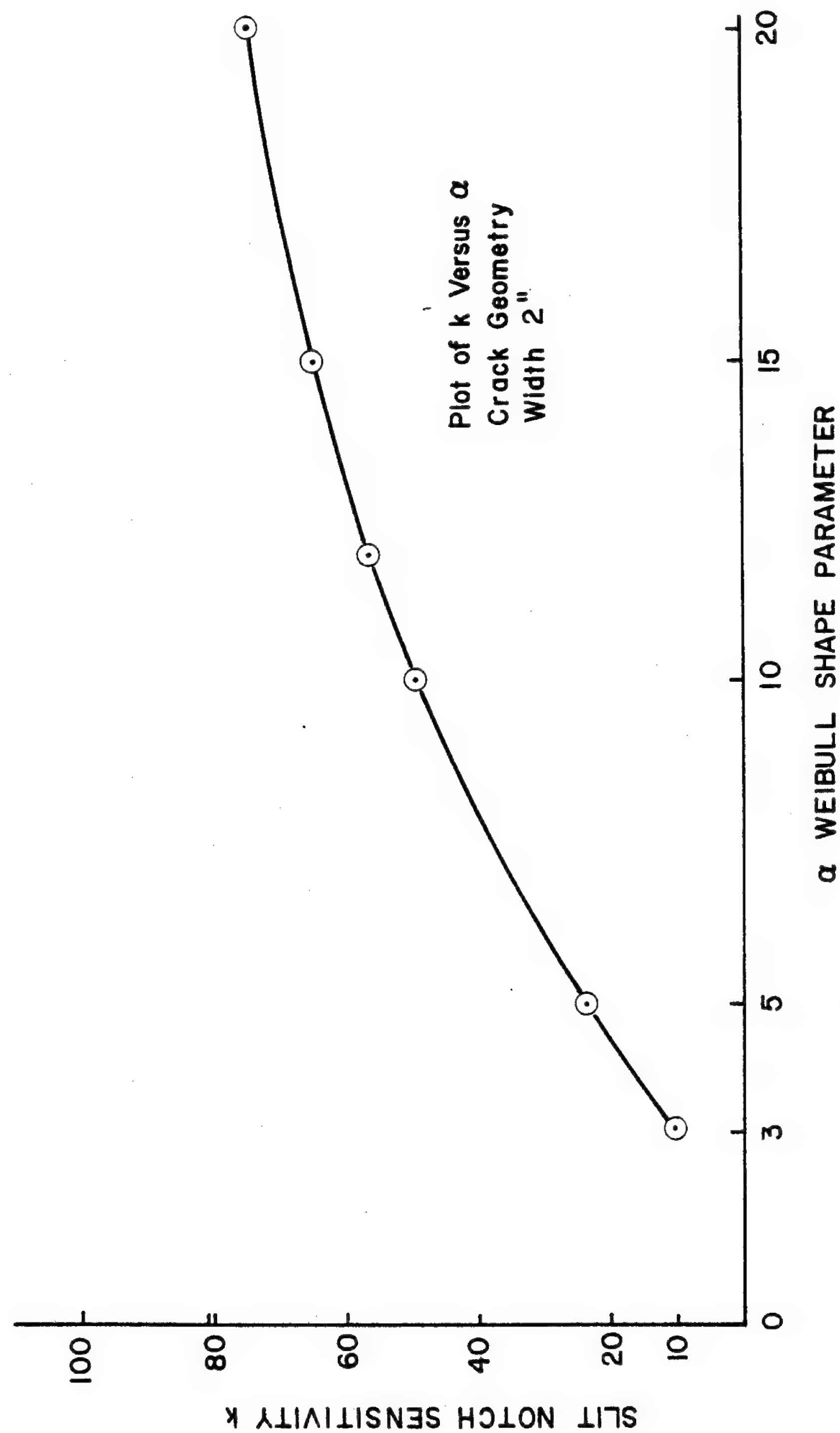

FIGURE 5

k CALCULATED AS A FUNCTION OF α FROM NUMERICAL INTEGRALS

1.4 Conclusions and Recommendations

I. The results of the Weibull calculation are conservative. Compare the notch sensitivity constants with experiment [Whitney, Kim 1976] for a $[90/0/\pm 45]_s$ Graphite/Epoxy:

Notch Geometry	CALCULATED USING α=20		EXPERIMENTAL (Curve Fit)	
	m	k	m	k
Hole	0.61	16.8	0.49	9.6
Slit	0.84	75.3	0.39	19.4

The higher values of k and m calculated by Weibull integrals predict a more rapid degradation of notched strength with larger notch size than is actually experienced. For circular holes, this conservatism is on the order of a design safety factor, with calculated notched strengths on the order of 1/1.5 to 1/2 lower than seen experimentally. For slit notches the calculation is extremely conservative. One explanation for this is that the theoretical stress profile, even our altered form 1.19 is more strenuous than that actually experienced. The model is an atomically thin crack; any machining takes the form of a slot.

Other reasons for conservatism on both circular and slit notches are: The volume of integration is the highest risk volume (highest stresses). Also, the Weibull theory

is a weak link theory (see appendix B) where failure of any volume brings about failure of the whole. This is a correct assumption for very brittle materials such as ceramics, but is somewhat conservative for composites.

<u>II</u> The notch sensitivity constants k, m are <u>volume dependent</u>. If one chooses a larger volume, the Weibull calculation predicts different notch sensitivity constants m,k (at a given α). For example:

GEOMETRY	CALC. USING α=10, W=2"		CALC. USING α=10, W=4"	
	m	k	m	k
Hole	0.52	9.7	0.61	6.1
Slit	0.72	49.3	0.76	37.6

This suggests that the sample volume in a notch-size series should be kept as constant as possible and practical. One suggestion for correcting between various volumes used experimentally is to define a standard volume, and scale experimental results to the standard by multiplying by the ratio of the theoretical calculations for the volumes. That is, if we let

$S = \sigma_N/\sigma_o$ experimental, at volume 2

$S^* = \sigma_N/\sigma_o$ experimental, "shifted" to standard volume 1

$I_2 = \sigma_N/\sigma_o$ calculated for volume 2 at given α

$I_1 = \sigma_N/\sigma_o$ calculated for volume 1 at given α

We let the change in volume be represented by a varying width "w." Thus,

$$S^* = S\left(\frac{I_1}{I_2}\right) \qquad 1.23$$

Corrections of this type have been modest, on the order of a few percent.

III The notch sensitivity of a material increases with increasing α. A high α value (from unnotched tensile tests) implies a very tight (narrow) distribution of inherent flaws. The rapid strength degradation with notch size for a material with high α implies that these inherent flaws are small. Due to the inherent "perfection" of the material, addition of a notch has a dramatic effect on strength.

Materials with low α thus have lower notch sensitivites; however, this is a mixed blessing. Low α values imply a wide distribution of various size inherent flaws. The addition of the notch (one more flaw) does not have as much impact on strength. But with low α, the spread of data becomes large. Designing to a given degree of reliability, one is forced to accept very low stresses.

This is due to the equation relating reliability "R"(σ) to α and β for an unnotched specimen.

$$\ln \sigma = \frac{1}{\alpha} \ln[-\ln R(\sigma)] + \ln \beta \qquad 1.24$$

where σ is the design allowable for the desired reliability R. (see Appendix A Part II) For example, for

$\beta = 5{,}000$

R = .9999 (Probability of failure = 1/10,000)

α	σ allowable	$\bar{\sigma}$ (average)
10	1990	4760
30	3680	4920

The low α material can use only a small percentage of its average unnotched failure stress.

IV The assumed form 1.13 for the variation of ineffective length with notch size used in the point-stress failure model is confirmed by the Weibull integrals. The form was a matter of convenient assumption, and is now satisfactorily proven by the integral results. See Tables 1 and 2.

V Future Work Areas

The data base for notch sensitivity constants k and m and tensile data scatter α needs to be expanded. This data allows a fuller exploration of the brittle failure theory, and also serves the designer.

The material constants k, m need to be related for various notch geometries. This could be obtained through a study of the stress solutions for general notch shapes, and these solutions decayed to the specific geometries of interest-hole, crack, etc.

Part 2 Biaxial Tension

2.1 Weibull Theory

A numerical integration can be used to predict basically conservative notched strengths for a material subjected to biaxial tension. The geometry studied was that of a circular hole in an infinite plate, and the material was assumed isotropic.

The direction of failure cannot be presumed as it was in the case of uniaxial tension. We assume that the most important stress is the opening (mode 1) stress σ_θ . See Figure 6. Under application of stresses σ_x , σ_y far away from the hole,* the tangential, normal, and shear stress shown in Fig. 6 are given by [Savin]

$$\sigma_\rho = \frac{\sigma_x}{2}\,[(1-\rho^2) + (1-4\rho^2+3\rho^4)\cos 2\theta] \qquad 2.1$$

$$+ \frac{\sigma_y}{2}\,[(1-\rho^2) - (1-4\rho^2+3\rho^4)\cos 2\theta]$$

$$\sigma_\theta = \frac{\sigma_x}{2}\,[(1+\rho^2) - (1+3\rho^4)\cos 2\theta \qquad 2.2$$

$$+ \frac{\sigma_y}{2}\,[(1+\rho^2) + (1+3\rho^4)\cos 2\theta]$$

---

* Should the original problem have a non-zero τ'_{xy} component, this may be reduced to a biaxial stress state by rotating through an angle β, where

$$\beta = \frac{1}{2}\tan^{-1}\left(\frac{2\tau'_{xy}}{\sigma'_x-\sigma'_y}\right)$$

The "prime" quantities are those of the original problem.

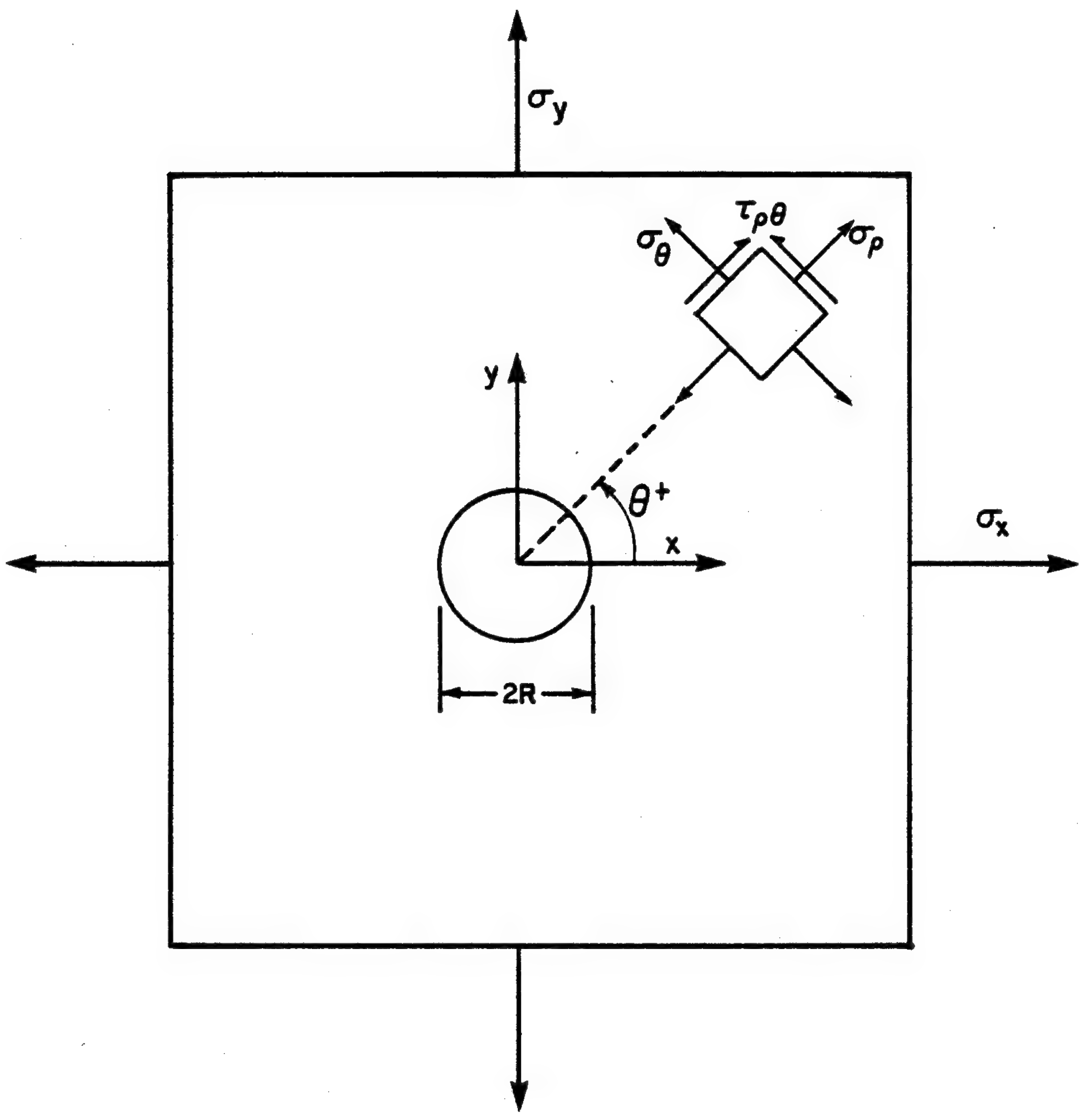

FIGURE 6

STRESS AND COORDINATE DEFINITIONS

$$\tau_{\rho\theta} = -\frac{\sigma_x}{2}[1+2\rho^2-3\rho^4]\sin 2\theta + \frac{\sigma_y}{2}[1+2\rho^2-3\rho^4]\sin 2\theta \qquad 2.3$$

where $\rho = \dfrac{R}{\sqrt{x^2+y^2}}$

Note that as the radius $R \to 0$, all ρ terms vanish, and we are left with the standard stress transformation (rotation of axis) formulae. [Crandall & Dahl]

The Weibull integrals used will be for the <u>opening mode</u> stress σ_θ . For a reliability function as in uniaxial tension. (See equation 1.5, 1.6)

$$R(\sigma) = e^{-B(\sigma)} \qquad 2.4$$

where

$$B(\sigma) = \int_V \left(\frac{\sigma_\theta}{\beta}\right)^\alpha dV \qquad 2.5$$

We again choose a high risk volume; for biaxial tension, however, we must investigate a number of rotation angles θ. The volume has length L, width δ, and plate thickness h. See Figure 7. The length L is chosen to be long enough to avoid finite width correction effects. The width δ is chosen "small" so that the stress is constant throughout the integration volume.

We assume that X is the principal loading axis, and divide equation 2.2 through by σ_x :

$$\frac{\sigma_\theta}{\sigma_x} = \frac{r+1}{2}(1+\rho^2) + \frac{r-1}{2}(1+3\rho^4)\cos 2\theta \qquad 2.6$$

where $r = \frac{\sigma_y}{\sigma_x}$ = LOADING RATIO

Evaluating equation 2.5 for $B(\sigma)$, we use equation 2.6, and the fact that the volume element $dV = \delta h\, d\zeta$ where $\zeta = \sqrt{x^2+y^2}$.

$$B(\sigma_x) = h\delta \int_R^L \left(\frac{\sigma_\theta}{\beta}\right) d\zeta \qquad 2.7a$$

$$= h\delta \frac{\sigma_x^\alpha}{\beta^\alpha} \int_R^L \left|\frac{r+1}{2}\left(1+\left(\frac{R}{\zeta}\right)^2\right) + \frac{r-1}{2}\left(1+3\left(\frac{R}{\zeta}\right)^4\right)\cos 2\theta\right]^\alpha d\zeta$$

2.7b

We combine the integral results and the material shape constant β into a single shape constant for notched biaxial failure

$$\boxed{\beta_{BN} = \beta\left(h\delta \int_R^L \left[\frac{r+1}{2}\left(1+\left(\frac{R}{\zeta}\right)^2\right) + \frac{r-1}{2}\left(1+3\left(\frac{R}{\zeta}\right)^4\right)\cos 2\theta\right]^\alpha d\zeta\right)^{-1/\alpha} \qquad 2.8}$$

This is useful, as we can relate the scale parameter to the average notched failure strength by using (See Appendix A.I)

$$\overline{\sigma}_{BF} = \beta_{BF}\,\Gamma(1+1/\alpha) \qquad 2.9$$

Note that equation 2.6 has inherently unknown constants such as δ within it; by normalizing the notched strength to an equivalent volume unnotched strength, we can get

rid of the unknown constants. Since we do have an isotropic material, we can compute an unnotched biaxial strength based also on σ_θ stresses. Simply let $\rho \to 0$ in equation 2.8, (no hole)

$$\beta_B = \beta\left(h\delta L[\frac{r+1}{2} + \frac{r-1}{2}\cos 2\theta]^{\alpha}\right)^{-1/\alpha} \qquad 2.10$$

Again, the scale parameter is related to the average unnotched strength by

$$\bar{\sigma}_B = \beta_B\,\Gamma(1+1/\alpha) \qquad 2.11$$

where $\bar{\sigma}_B$ = the average value of σ_x on failure.

We now ratio unnotched strength to notched strength to eliminate some of the unknown constants:

$$\boxed{\frac{\sigma_{BN}}{\sigma_B} = \frac{\left(\int_R^L [\frac{r+1}{2}(1+(\frac{R}{\zeta})^2) + \frac{r-1}{2}(1+3(\frac{R}{\zeta})^4)\cos 2\theta]^{\alpha} d\zeta\right)^{-1/\alpha}}{L^{-1/\alpha}[\frac{r+1}{2} + \frac{r-1}{2}\cos 2\theta]^{-1}} \qquad 2.12}$$

NOTE that this assumes that α stays constant for the unnotched and notched strength tests. All items in equation 2.12 may now be calculated for various angles θ at a given loading ratio r. The angle θ which provides the lowest $\frac{\sigma_{BN}}{\sigma_B}$ indicates the failure location (direction) and value.

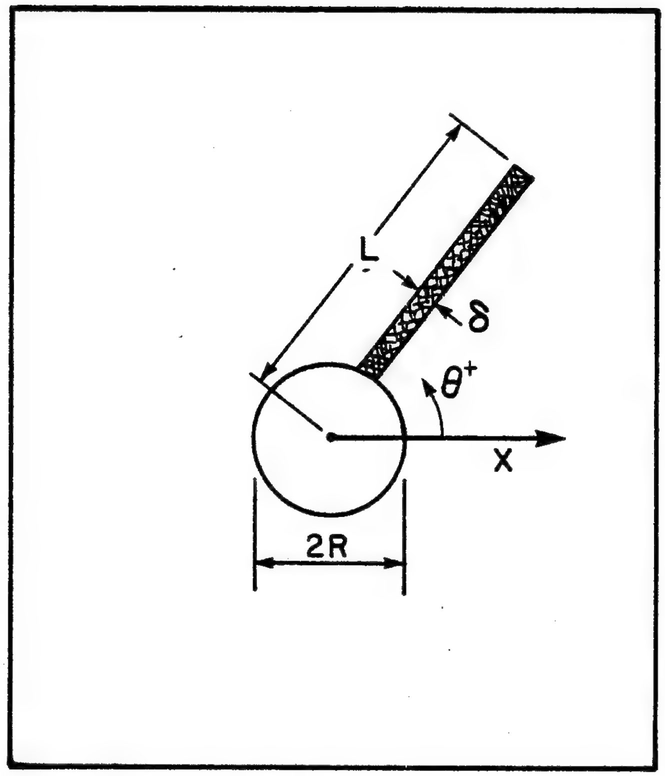

PLATE THICKNESS = h

FIGURE 7

VOLUME OF INTEGRATION FOR NUMERICAL INTEGRAL

2.2 Numerical Integral Results (Circular hole)

The numerical integration necessary to evaluate the average notched strength was performed by the IMSL routine DCADRE. The numerical values used in this study were:

hole radius R = 0.5"

integration length L = 5."

Weibull shape parameter α = 10.

The loading ratio $r = \frac{\sigma_y}{\sigma_x}$ was varied from 0 to 10. The result for $r = 0 = \sigma_y$ correctly predicts the uniaxial strength.

Table 3

r	θ_{fail}	σ_{BN}/σ_B
0	90.	0.568
0.1	90.	0.584
0.5	90.	0.659
1.	ANY	0.764
2.	0.	0.659
3.	0.	0.626
5.	0.	0.602
10.	0.	0.584

The failure direction is thus seen to "snap through" so that it is always perpendicular to the

largest applied load. This helps avoid awkwardness in the nature of σ_B , the unnotched strength; if θ were not 0° or 90°, there is no assurance that the unnotched sample would ever fail in the direction θ. Thus, biaxial unnotched strength σ_B is equivalent to the uniaxial unnotched strength "σ_o" . At r = 1 we reach a state of equal stresses for any θ, so the failure may propagate in any direction.

The addition of the second load axis is also seen to contribute to strength. This is explained as follows. With a single axis load, say σ_x only, there are high tensile stresses at $\theta = 90°$. The addition of σ_y provides compressive stresses which partially offset the high tensile stresses in this critical region at $\theta = 90°$.

The numerical integrals were performed for a series of hole sizes and compared with data from a quasi-isotropic graphite/epoxy laminate $[0/\pm45/90]_s$ [Daniel]. The results are shown first for uniaxial, then biaxial strengths. The α value is not known, but is probably in the 20-30 range from experience.

Table 4

Hole Diameter (2R) Inch	Exper. Notched Str. σ_N , MPa	Exper. σ_N/σ_o	Calculated σ_N/σ_o	
			$\alpha = 10$	$\alpha = 25$
1.0	219	0.436	0.568	0.429
0.75	231	0.460	0.584	0.434
0.50	255	0.508	0.608	0.441
0.25	276	0.550	0.651	0.454

The integrals for the calculated σ_N/σ_o used integration $L = 5"$, loading ratio r = 0. No finite width correction factors were used in experimental or calculated results. The experimental results are bracketed by calculated values for $\alpha = 10$ and $\alpha = 25$. The caculations for $\alpha = 25$ are conservative.

Table 5

Hole Diameter 2R Inch	Exper. Biax. Str. σ_{BF} , MPa	Exper. σ_{BF}/σ_B	Calculated σ_{BF}/σ_B	
			$\alpha = 10$	$\alpha = 25$
1.0	276	0.550	0.764	0.621
0.75	280	0.558	0.785	0.629
0.50	320	0.637	0.814	0.639
0.25	366	0.729	0.862	0.657

We see for the first time a non-conservative failure prediction σ_{BF}/σ_B with the calculation up to 13% higher than experiment. However, the data scatter in the experimental results is large, and the results somewhat inconclusive. (Experimental result $\sigma_{BF}/\sigma_B = 0.558$ is average of two points 0.614, 0.543.)

2.3 Conclusions and Recommendations

I The calculated biaxial failure loads are generally conservative versus experimental data. That is, the numerical Weibull integrals predict a more severe strength degradation in a notched sample than we actually experience. The few cases of non-conservatism (see Table 5) are inconclusive due to a scarcity of data. The conservatism of calculation occurs as the calculations use the highest risk volume with a "weak link" theory. In actuality, there would be some "give" to the material.

II The Weibull biaxial integration results accurately predict the strengthening effect of the stress along the second axis. This strengthening effect is due to compressive stresses from one axis loading partially cancelling the tensile stresses from the other axis loading.

Further, the Weibull integration predicts within 15% the magnitude of the strengthing effect for equal axis loading: (from Tables 4, 5)

Hole Diameter 2R Inch	Experimental σ_{BF}/σ_o ($\sigma_x=\sigma_y$)	Calculated σ_{BF}/σ_o ($\sigma_x=\sigma_y$)	
		$\alpha=10$	$\alpha=25$
1.00	1.26	1.35	1.45
0.75	1.21	1.34	1.45
0.50	1.25	1.34	1.45
0.25	1.33	1.32	1.45
AVG	1.26	1.34	1.45

where σ_{BF}/σ_o is the biaxial strength with $\sigma_x = \sigma_y$ divided by the uniaxial strength.

III The Weibull integration results for the failure direction agree with the limited amount of experimental data. Experimentally, the biaxial samples filed perpendicular to the stronger load, with inconclusive results for the equal biaxial loading case. [Daniel] Some of the materials tested in reference [Daniel] were anisotropic, and no predictive calculations have been performed.

IV Future Work Areas:

Fracture of anisotropic materials in biaxial tension can be similarly addressed by numerical evaluation of Weibull integrals. The failure direction of such materials can be predicted, along with some measure of strength reduction in the presence of a notch. These calculations should be supported by more biaxial failure data. Having done this for the biaxial load case, it should be simple to do the uniaxial load case as a reduction.

Experience needs to be gained in how to properly test samples in biaxial loading. The simple tube pressurization test is, of course, not suitable for notched geometries. The existing literature dealing with this problem is very small.

References

1. S. H. Crandall, N. C. Dahl, An Introduction to the Mechanics of Solids, McGraw-Hill, c 1959, pp. 133-136.

2. I. M. Daniel, Biaxial Testing of Graphite/Epoxy Composites Containing Stress Concentrations, Part I, AFML TR-76-244, 1976.

3. H. J. Konish, J. M. Whitney, "Approximate Stresses in an Orthotropic Plate Containing a Circular Hole," J. Composite Materials, Vol. 9, April 1975.

4. P. C. Paris, G. C. Sih, Fracture Toughness Testing and its Applications, ASTM STP 381, American Society for Testing and Materials, 1965, pp. 84-113.

5. R. E. Peterson, Stress Concentration Factors, Wiley-Interscience, c 1974, pp. 110-112.

6. R. B. Pipes, R. C. Wetherhold, J. W. Gillespie, Jr., "Notched Strength of Composite Materials," J. Composite Materials, Vol. 13, April 1979.

7. R. B. Pipes, J. W. Gillespie, Jr., R. C. Wetherhold, "Superposition of the Notched Strength of Composite Laminates," Polymer Engineering and Science, Vol. 19, December, 1979.

8. G. N. Savin, Stress Concentration Around Holes, transl. by W. Johnson, Pergamon Pres, c 1961.

9. J. M. Whitney, R. J. Nuismer, "Stress Fracture Criteria for Laminated Composites Containing Stress Concentration," J. Composite Materials, Vol. 8, July, 1974.

10. J. M. Whitney, R. Y. Kim, Effect of Stacking Sequence on the Notched Strength of Composite Laminates, AFML TR 76-177, 1976.

APPENDIX A: Mathematical Development of the Weibull Distribution

I Mean and Standard Deviation

The generally used form for a cumulative distribution function (or simply "distribution function") F is:

$$F(\sigma) = P(\sigma_s \leq \sigma) \qquad \text{I.1}$$

where σ_s is a random variable*, P denotes probability, and σ is the particular value of interest. For a set of data which obeys a Weibull distribution, the distribution function is:

$$F(\sigma) = 1 - \exp[-(\sigma/\beta)^{\alpha}] \quad \sigma \geq 0 \qquad \text{I.2}$$

where exp is exponential function

β is the scale or location parameter

α is the shape parameter

Mathematicians often use the equivalent notation

$$\sigma \overset{d}{\sim} W(\beta,\alpha) \qquad \text{I.3}$$

to show that σ obeys a Weibull distribution.

We are often interested in the reliability, i.e. the probability that the random variable (outcome of an experiment) exceeds the particular value of interest. This reliability is expressed as

*A "random variable" is simply the outcome of an experiment. For example, the value shown on a fair die after a die toss is a random variable.

$$R(\sigma) = P(\sigma_s \geq \sigma) \qquad \text{I.4}$$

or

$$R(\sigma) = 1 - F(\sigma)$$

and, for a Weibull distribution, is given by:

$$R(\sigma) = \exp[-(\sigma/\beta)^{\alpha}] \quad \sigma \geq 0 \qquad \text{I.5}$$

We now proceed to find the <u>mean</u> or average in terms of the parameters α,β. To get the probability density function (p.d.f.) $p(\sigma)$ of a distribution, take the derivative as follows:

Since p.d.f. $p(\sigma)d\sigma = P[\sigma - d\sigma \leq \sigma_s \leq \sigma + d\sigma]$ I.6

$$p(\sigma) = \frac{d}{d\sigma}[F(\sigma)] \qquad \text{I.7}$$

Using the Weibull form for the distribution function F(I.1),

$$p(\sigma) = \sigma^{\alpha-1}\,\beta^{-\alpha}\,\exp[-(\sigma/\beta)^{\alpha}] \qquad \text{I.8}$$

The formula for obtaining the mean is:

$$E[\sigma_s] = \overline{\sigma} = \int_0^{\infty} \sigma\, p(\sigma)\, d\sigma \qquad \text{I.9}$$

where $\overline{\sigma}$ denotes the average, and E is the expected value operator. The form for the Weibull p.d.f. I.8 is substituted into I.9. The ensuing integration uses the change of variable

$$x = \beta^{-\alpha}\,\sigma^{\alpha}$$

which does not change the limits of integration. The final expression is

$$\overline{\sigma} = \beta \int_0^\infty x^{(1+1/\alpha)-1}\, e^{-x}\, dx \qquad \text{I.10}$$

This form is identical to the Gamma Function Γ, a table look-up in mathematical handbooks. Thus we have

$$\overline{\sigma} = \beta\Gamma(1+1/\alpha) \qquad \text{I.11}$$

For $\alpha \geq 1$, $0.886 \leq \Gamma \leq 1.$, so that $\overline{\sigma} \leq \beta$; or the average is less than the scale parameter due to the skewness of the distribution.

The standard deviation for a Weibull distribution may be found similarly from the basic definition:

$$\text{St.Dev.} \quad S = \{E(\sigma_s^2] - (E[\sigma_s])^2\}^{\frac{1}{2}} \qquad \text{I.12}$$

When the definition I.12 is used with the p.d.f. definition I.8, it yields:

$$S = \{\beta^2[\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)]\}^{\frac{1}{2}} \qquad \text{I.13}$$

II Statistical Inference of Weibull Parameters

Given a set of data from an experiment, we would like to predict point extimates α and $\hat{\theta}$ for the entire (infinite) population. One way to do this is by use of logarithms; this method should be used only if a digital computer or programmable calculator is not available. The second method is that of solution of the maximum likelihood equations; this demands use of a computer (or programmable calculator).

Logarithm Method

The data are ranked from lowest to highest, and a ranking statistic is used. The rank can be simple, mean, median, hazard, or Hazen; but there is a preference for the median rank. This median rank is applied to the ranked data as

$$P_j \quad \text{from tables}$$

$$\text{or} \qquad P_j \simeq \frac{j - 0.3}{n + 0.4} \qquad \text{II.1}$$

which gives a series of j ranks of the n total data points. This rank statistic is used as an approximation to the distribution function F (see I.1). It is helpful to draw a graph of data value (abcissa) versus rank (ordinate).

We proceed to manipulate the formula I.2 into a convenient linear form. Since $P(\sigma_s \leq \sigma) = F(\sigma)$,

$$1-F = \exp[-(\sigma/\beta)^{\alpha}] \qquad \text{II.2}$$

Taking natural logarithms,

$$\ln(1-F) = -(\sigma/\beta)^{\alpha} \qquad \text{II.3}$$

Clearing the minus sign and again taking natural logarithms,

$$\ln(-\ln(1-F) = \alpha \ln(\sigma/\beta) \qquad \text{II.4a}$$

or

$$\ln\sigma = \frac{1}{\alpha} \ln[-\ln(1-F)] + \ln\beta \qquad \text{II.4b}$$

The equation II.4b is in linear form for a linear squares analysis of data points $\ln\sigma$ versus $\ln[-\ln(1-\text{Rank})]$. The slope is $1/\alpha$, the intercept is $\ln\beta$.

Note that the form II.4b is also usable to calculate an allowable σ for a given structural reliability $R = 1-f$, once the α and β are known.

Problems with logarithm method: The α,β found by linear regression on II.4b are point estimates. To obtain meaningful interval estimates of say α requires us to assume that the least squares statistics $1/\alpha$ and $\ln\hat{\beta}$ are normally distributed. This assumption seems doubtful at best. The logarithm method also has difficulty handling censored data points, which can occur in fatigue experiments

(as run-out, etc.).

Maximum Likelihood Method

This method has a sound theoretical basis and allows interval estimates to be made for $\alpha, \hat{\sigma}$. However, it demands solution of a very non-linear equation.

The maximum likelihood estimator $\hat{\theta}_n$ for a set of n random variables X_i is given by the solution of:

$$L(\theta) = f(x_{.};\theta) f(x_2;\theta) \ldots f(x_n;\theta) \qquad \text{II.5}$$

$$\text{MAX } L(\theta) = \prod_{i=1}^{n} \frac{\alpha}{\beta} \left(\frac{x_i}{\beta}\right)^{\alpha-1} \exp\left[-\left(\frac{x_i}{\beta}\right)^{\alpha}\right] \qquad \text{II.6}$$

where f is the p.d.f. of X, and θ is the true population arameter, and $L(\theta)$ reaches a maximum at $\hat{\theta}$.[A.1] maximization equation II.6 acts to maximize the likelihood of "legitimizing" the outcome of a given experiment.[A.2]

The maximum likelihood equations for $\hat{\alpha}, \hat{\beta}$ are given by:

$$k(\hat{\alpha}) = 0 = \frac{\sum_{i=1}^{n} X_i^{\hat{\alpha}} \ln X_i}{\sum_{i=1}^{n} X_i^{\hat{\alpha}}} - \frac{1}{\hat{\alpha}} - \frac{\sum_{i=1}^{n} \ln X_i}{n} \qquad \text{II.7}$$

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^{n} X_i^{\hat{\alpha}}\right)^{1/\hat{\alpha}} \qquad \text{II.8}$$

The $\hat{\alpha}, \hat{\beta}$ are point estimates to the true population parameters α, β. [A.3] In practice, equation II.7 is solved for $\hat{\alpha}$ and

the result used in II.8 to get $\hat{\beta}$. For small sample sizes, the $\hat{\alpha}, \hat{\beta}$ estimates from equations II.7, II.8 are <u>biased</u>. Fortunately, there are correction factors available which depend only on n, the number of data points [A.4]. These correction factors B(n), shown below, thus provide unbiasing so that

$$E[B(n)\hat{\alpha}] = \alpha \qquad \text{II.9}$$

TABLE A.1

Unbiasing Factors B(n) for MLE of α

n	5	6	7	8	9	10	11	12	13	14	15	16
B(n)	.669	.752	.792	.820	.842	.859	.872	.883	.893	.901	.908	.914
n	18	20	22	24	26	28	30	32	34	36	38	40
B(n)	.923	.931	.938	.943	.947	.951	.955	.958	.960	.962	.964	.966
n	42	44	46	48	50	52	54	56	58	60	62	64
B(n)	.968	.970	.971	.972	.973	.974	.975	.976	.977	.978	.979	.980
n	66	68	70	72	74	76	78	80	85	90	100	120
B(n)	.980	.981	.981	.982	.982	.983	.983	.984	.985	.986	.987	.990

Interval estimates for α and β may be obtained for any desired level γ of significance desired. [A.4] The intervals can be one-sided or two-sided.

References

[A.1] N. R. Mann, R. E. Schafer, N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data, John Wiley & Sons, New York, 1974.

[A.2] William Mendenhall and Richard Schaeffer, Mathematical Statistics with Applications, Duxbury Press, 1973.

[A.3] N. R. Mann, et. al., op. cit.

[A.4] D.R.Thoman, L. J. Bain, C.E. Antle, "Inferences on the Parameters of the Weibull Distribution," Technometrics, Vol. 11, No. 3, 1969.

APPENDIX B: Physical Development of the Weibull Distribution

The Weibull distribution was developed as a distribution of wide utility for fitting data. It has a particular physical significance in treating fracture data of brittle material. The central thesis of this concept is that the strength of a material cannot be expressed by a single number. The risk of breakage may vary throughout the material; the break site need not be the site at which the ultimate stress is first reached.

If the material is viewed as a series of n volume elements and the individual element failure probabilities are S_1, S_2,...S_n, then the survival of the whole (no failure) depends on the survival of all n elements. i.e.

$$1-S = (1-S_1)(1-S_2) \ldots (1-S_n) \qquad 1.1$$

NOTE the "weak link" effect implied here; failure of any subpart brings failure of the whole. For infinitely small elements, the probability of failure in any one given element becomes vanishingly small, thus, in 1.1 we multiply the right-hand-side and omit the higher order terms:

$$S = S_1 S_2 S_3 \ldots S_n \qquad 1.2$$

The probability of an individual volume failure is proportional to the volume; letting F(x) be the distribution

function for the infinitely small volume,

$$S_i = F_i(\sigma)\ dV \qquad i = 1,2,\ldots n \qquad 1.3$$

where F_i is finite and dV is infinitely small.

The value of σ, which could represent stress or strain, is in general a function of position, i.e. $\sigma = \sigma(\vec{X})$. If σ_{MAX} represents the maximum stress at any point in the material, then the local stress may be considered a product of σ_{MAX} and a geometric locus function $t(\vec{x})$ so that

$$\sigma = \sigma_{MAX} \cdot t(\vec{x}) \qquad 1.4$$

From 1.1, we obtain generally that

$$\log(1-S) = \sum_{\mu=1}^{n} \log(1-S_\mu) \qquad 1.5$$

As the number of volume elements, n, increases indefinitely, S_μ converges to zero, so that

$$\log(1-S_\mu) = -S_\mu \qquad 1.6$$

Using equation 1.5 as $n \to \infty$,

$$\log(1-S) = -\lim_{n\to\infty} \sum^{n} S_\mu \qquad 1.7$$

In the limit, using 1.3 with 1.7,

$$\log(1-S) = -\int_V F(\sigma)dV \qquad 1.8$$

If we define for simplicity

$$B = -\int_V F(\sigma)dV \qquad 1.9$$

Then the distribution function for the entire body

$$F(\sigma) = 1 - e^{-B(\sigma)} \qquad 1.10$$

This F gives the probability of failure for the outcome of any particular experiment as a function of the applied stress σ.

Weibull [B.1] chooses an elementary distribution function which satisfies equation 1.1

$$F(\sigma) = \left(\frac{\sigma-\sigma_u}{\beta}\right)^{\alpha} \qquad 1.11$$

where σ_u is the value below which no volume will ever fail (usually taken to be zero).

Reference

[B.1] W. Weibull, "The Phenomenon of Rupture in Solids," Ingeniörs Vetenskaps Akademien Landlingar Nr. 153, Stockholm, 1939, (In English.)